

# Extrapolation formulae by parameterizations and their comparative analysis.

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## Abstract

*Here we propose a new form of extrapolation formula using a parametric assumption and try to generalize this (as much as possible) by selecting different values of an arbitrary constant for generating specific outputs corresponding to some inputs in specific positions. That is we use a set of arguments (values of the independent variable) with corresponding functional values (values of the dependent variable). The relative errors in estimated values based on exact values (expressed as a percentage) will be used as the scale of the accuracy that will measure the fluctuation of the approximate values from the exact values. Also, we will provide a comparative analysis, among three fundamental assumptions of parameterization using suitable examples. Finally a conclusive discussion will be given.*

## 1. Introduction

**Extrapolation** is a mathematical technique of inferring unknown from the known. It attempts to predict future data by relying on historical data, such as estimating the size of a population a few years from now on the basis of current population size and its rate of growth. Extrapolation may be valid where the present circumstances do not indicate any interruption in the long-established past trends. However, a straight line extrapolation (where a short-term trend is believed to continue far into the future) is fraught with risk because some unforeseeable factors almost always intervene.

Extrapolation [2] is the process of estimating or predicting the value  $y$  of an unknown [3] but observable function  $f$  at a point  $x_3$  outside the range of points where the value of  $y$  can be observed. If  $x_1$  is the minimum point at which the value of  $f$  has been observed and  $x_2$  is the maximum point, then  $x_3$  is outside this range (either  $x_3 < x_1$  or  $x_3 > x_2$ ). We use the form for  $f$  that we have fitted on the range  $[x_1, x_2]$  to estimate or extrapolate the value  $y = f(x_3)$ . If we have fitted  $f$  such that it is linear [7] on the range  $[x_1, x_2]$ , so that

$$f(x) = \frac{f(x_2)(x-x_1)-f(x_1)(x-x_2)}{x_2-x_1}$$

then we extend this line out to  $x_3$  and linearly extrapolate  $y$  as  $f(x_3)$ . The above concept can be generalized. In Figure-01, the values of  $y$  at  $x_1, x_2, x_3$  and  $x_4$  are given and the value of  $y$  at  $x_5$  is to be extrapolated.

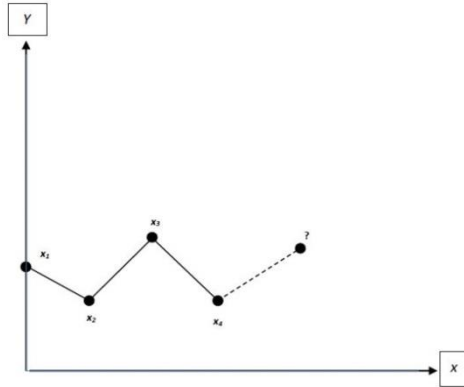


Figure-01 (Extrapolation)

Since the numerical solution of ordinary differential equations using numerical methods are sometimes not satisfactory because the results obtained are not accurate, we need to use extrapolation to obtain better accuracy. There is a variety of approaches available for extrapolating the numerical solution of initial value problems, including polynomial extrapolation, rational extrapolation and exponential extrapolation.

**Initial conditions** are given values of the function and/or its derivative(s) at a specific point. The exact solution and approximate values of dependent variable depend on the initial value or values. A differential equation gives rise to a family of solution curves [4], each corresponding to one value or set of values of the initial condition or conditions. For a variety of initial values these solution curves may diverge from one another, which have an important numerical significance.

Suppose for instance, if an initial value is read into the computer with some round off error, then even if all subsequent calculations are precise and no truncation errors occur, the computed solution will be wrong. An error made in the beginning has the effect of selecting the wrong curve from the family of solution curves. Any minute error made in the beginning may be responsible for an eventual complete loss of accuracy. This phenomenon is not restricted to errors made at the first step, because each point in the numerical solution can be interpreted as the initial value or values for succeeding points.

## 2. Derivation of Extrapolation formulae

Consider the parametric formula [9] for finding the extrapolation formulae

$$y_{n+k} = A_0y_n + A_1y_{n-1} + A_2y_{n-2} + A_3y_{n-3} + A_4y_{n-4} + h[B_0y'_n + B_1y'_{n-1} + B_2y'_{n-2} + B_3y'_{n-3} + B_4y'_{n-4}] \quad (1)$$

Here  $A_0, A_1, A_2, A_3, A_4, B_0, B_1, B_2, B_3$  &  $B_4$  are all parameters [6] and  $h$  is the spacing or increment in arguments. Using  $h = 1$  in Equation (1) and letting  $y(x) = x^n; (n = 0,1,2,3,4,5)$  consecutively in Equation (1) we get

$$A_0 + A_1 + A_2 + A_3 + A_4 = 1 \quad (2)$$

$$-A_1 - 2A_2 - 3A_3 - 4A_4 + B_0 + B_1 + B_2 + B_3 + B_4 = k \quad (3)$$

$$A_1 + 4A_2 + 9A_3 + 16A_4 - 2B_1 - 4B_2 - 6B_3 - 8B_4 = k^2 \quad (4)$$

$$-A_1 - 8A_2 - 27A_3 - 64A_4 + 3B_1 + 12B_2 + 27B_3 + 48B_4 = k^3 \quad (5)$$

$$A_1 + 16A_2 + 81A_3 + 256A_4 - 4B_1 - 32B_2 - 108B_3 - 256B_4 = k^4 \quad (6)$$

$$-A_1 - 32A_2 - 243A_3 - 1024A_4 + 5B_1 + 80B_2 + 405B_3 + 1280B_4 = k^5 \quad (7)$$

Treating Equation (2) as a linear relationship between  $A_0, A_1, A_2, A_3$  and  $A_4$ , and solving the above system of Equations (2)-(7) for  $B_0, B_1, B_2, B_3$  and  $B_4$  in terms of these, we get

$$A_0 = 1 - A_1 - A_2 - A_3 - A_4 \quad (8)$$

$$B_0 = (1/720)[6k^5 + 75k^4 + 350k^3 + 750k^2 + 720k + 251A_1 + 232A_2 + 243A_3 + 224A_4] \quad (9)$$

$$B_1 = (1/360)[-12k^5 - 135k^4 - 520k^3 - 720k^2 + 323A_1 + 496A_2 + 459A_3 + 512A_4] \quad (10)$$

$$B_2 = (1/60)[3k^5 + 30k^4 + 95k^3 + 90k^2 - 22A_1 + 16A_2 + 54A_3 + 32A_4] \quad (11)$$

$$B_3 = (1/360)[-12k^5 - 105k^4 - 280k^3 - 240k^2 + 53A_1 + 16A_2 + 189A_3 + 512A_4] \quad (12)$$

$$B_4 = (1/720)[6k^5 + 45k^4 + 110k^3 + 90k^2 - 19A_1 - 8A_2 - 27A_3 + 224A_4] \quad (13)$$

**Case-1:** Considering  $A_1 = A_2 = A_3 = A_4 = 0$

So,  $A_0 = 1$

$$B_0 = (1/720)[6k^5 + 75k^4 + 350k^3 + 750k^2 + 720k]$$

$$B_1 = (1/360)[-12k^5 - 135k^4 - 520k^3 - 720k^2]$$

$$B_2 = (1/60)[3k^5 + 30k^4 + 95k^3 + 90k^2]$$

$$B_3 = (1/360)[-12k^5 - 105k^4 - 280k^3 - 240k^2]$$

$$B_4 = (1/720)[6k^5 + 45k^4 + 110k^3 + 90k^2]$$

So, from Equation (1) we get

$$y_{n+k} = y_n + h[(1/720)[6k^5 + 75k^4 + 350k^3 + 750k^2 + 720k]y'_n + (1/360)[-12k^5 - 135k^4 - 520k^3 - 720k^2]y'_{n-1} + (1/60)[3k^5 + 30k^4 + 95k^3 + 90k^2]y'_{n-2} + (1/360)[-12k^5 - 105k^4 - 280k^3 - 240k^2]y'_{n-3} + (1/720)[6k^5 + 45k^4 + 110k^3 + 90k^2]y'_{n-4}] \quad (14)$$

**Case-2:** Considering  $A_0 = A_1 = A_3 = A_4 = 0$

So,  $A_2 = 1$

$$B_0 = (1/720)[6k^5 + 75k^4 + 350k^3 + 750k^2 + 720k + 232]$$

$$B_1 = (1/360)[-12k^5 - 135k^4 - 520k^3 - 720k^2 + 496]$$

$$B_2 = (1/60)[3k^5 + 30k^4 + 95k^3 + 90k^2 + 16]$$

$$B_3 = (1/360)[-12k^5 - 105k^4 - 280k^3 - 240k^2 + 16]$$

$$B_4 = (1/720)[6k^5 + 45k^4 + 110k^3 + 90k^2 - 8]$$

So, from Equation (1) we get

$$y_{n+k} = y_{n-2} + h[(1/720)[6k^5 + 75k^4 + 350k^3 + 750k^2 + 720k + 232]y'_n + (1/360)[-12k^5 - 135k^4 - 520k^3 - 720k^2 + 496]y'_{n-1} + (1/60)[3k^5 + 30k^4 + 95k^3 + 90k^2 + 16]y'_{n-2} + (1/360)[-12k^5 - 105k^4 - 280k^3 - 240k^2 + 16]y'_{n-3} + (1/720)[6k^5 + 45k^4 + 110k^3 + 90k^2 - 8]y'_{n-4}] \quad (15)$$

**Case-3:** Considering  $A_0 = A_1 = A_2 = A_3 = 0$

So,  $A_4 = 1$

$$B_0 = (1/720)[6k^5 + 75k^4 + 350k^3 + 750k^2 + 720k + 224]$$

$$\begin{aligned}
 B_1 &= (1/360)[-12k^5 - 135k^4 - 520k^3 - 720k^2 + 512] \\
 B_2 &= (1/60)[3k^5 + 30k^4 + 95k^3 + 90k^2 + 32] \\
 B_3 &= (1/360)[-12k^5 - 105k^4 - 280k^3 - 240k^2 + 512] \\
 B_4 &= (1/720)[6k^5 + 45k^4 + 110k^3 + 90k^2 + 224]
 \end{aligned}$$

So, from Equation (1) we get

$$\begin{aligned}
 y_{n+k} &= y_{n-4} + h[(1/720)[6k^5 + 75k^4 + 350k^3 + 750k^2 + 720k + 224]y'_n + (1/360)[-12k^5 - 135k^4 - 520k^3 - 720k^2 + 512]y'_{n-1} + (1/60)[3k^5 + 30k^4 + 95k^3 + 90k^2 + 32]y'_{n-2} + (1/360)[-12k^5 - 105k^4 - 280k^3 - 240k^2 + 512]y'_{n-3} + (1/720)[6k^5 + 45k^4 + 110k^3 + 90k^2 + 224]y'_{n-4}]
 \end{aligned} \tag{16}$$

### 3. Particular Forms

To apply the formulae derived above we need their particular forms. In this section we assign particular values of  $k$  &  $n$  for finding particular forms of extrapolation formulae.

Putting  $k = 1$  &  $n = 4$  in Equations (14), (15) and (16) respectively, after simplifying we get

$$y_5^{(1)} = y_4 + (h/720)[1901y'_4 - 2774y'_3 + 2616y'_2 - 1274y'_1 + 251y'_0]$$

$$y_5^{(2)} = y_2 + (h/80)[237y'_4 - 198y'_3 + 312y'_2 - 138y'_1 + 27y'_0]$$

$$y_5^{(3)} = y_0 + (h/144)[425y'_4 - 350y'_3 + 600y'_2 - 50y'_1 + 95y'_0]$$

Putting  $k = 3$  &  $n = 4$  in Equations (14), (15) and (16) respectively, after simplifying we get

$$y_7^{(1)} = y_4 + (h/80)[2877y'_4 - 7638y'_3 + 8712y'_2 - 4698y'_1 + 987y'_0]$$

$$y_7^{(2)} = y_2 + (h/720)[26125y'_4 - 67750y'_3 + 78600y'_2 - 42250y'_1 + 8875y'_0]$$

$$y_7^{(3)} = y_0 + (h/720)[26117y'_4 - 67718y'_3 + 78792y'_2 - 41258y'_1 + 9107y'_0]$$

Putting  $k = 6$  &  $n = 4$  in Equations (14), (15) and (16) respectively, after simplifying we get

$$y_{10}^{(1)} = y_4 + (h/10)[3483y'_4 - 11292y'_3 + 14328y'_2 - 8292y'_1 + 1833y'_0]$$

$$y_{10}^{(2)} = y_2 + (h/360)[125504y'_4 - 406016y'_3 + 515904y'_2 - 298496y'_1 + 65984y'_0]$$

$$y_{10}^{(3)} = y_0 + (h/18)[6275y'_4 - 20300y'_3 + 25800y'_2 - 14900y'_1 + 3305y'_0]$$

Putting  $k = 9$  &  $n = 4$  in Equations (14), (15) and (16) respectively, after simplifying we get

$$y_{13}^{(1)} = y_4 + (h/80)[129861y'_4 - 451494y'_3 + 600696y'_2 - 360234y'_1 + 81891y'_0]$$

$$y_{13}^{(2)} = y_2 + (h/720)[1168981y'_4 - 4062454y'_3 + 5406456y'_2 - 3242074y'_1 + 737011y'_0]$$

$$y_{13}^{(3)} = y_0 + (h/720)[1168973y'_4 - 4062422y'_3 + 5406648y'_2 - 3241082y'_1 + 737243y'_0]$$

Putting  $k = 12$  &  $n = 4$  in Equations (14), (15) and (16) respectively, after simplifying we get

$$y_{16}^{(1)} = y_4 + (h/5)[26178y'_4 - 94272y'_3 + 128808y'_2 - 78912y'_1 + 8258y'_0]$$

$$y_{16}^{(2)} = y_2 + (h/180)[942466y'_4 - 3393544y'_3 + 4637136y'_2 - 2840824y'_1 + 657286y'_0]$$

$$y_{16}^{(3)} = y_0 + (h/180)[942464y'_4 - 3393536y'_3 + 4637184y'_2 - 2840576y'_1 + 657344y'_0]$$

#### 4. Numerical Examples

I. Solve  $y' = \frac{dy}{dx} = 2e^x - y$  at  $x = 0.50, x = 0.70, x = 1.00, x = 1.30, x = 1.60$

With initial values  $y(0.00) = 2.0000, y(0.10) = 2.0100, y(0.20) = 2.0401, y(0.30) = 2.0907$  &  $y(0.40) = 2.1621$  provided by the analytical solution  $y_{Exact} = e^x + e^{-x}$

II. Solve  $y' = \frac{dy}{dx} = 3e^x + 2y$  at  $x = 0.50, x = 0.70, x = 1.00, x = 1.30, x = 1.60$

With initial values  $y(0.00) = 0.0000, y(0.10) = 0.3487, y(0.20) = 0.8113, y(0.30) = 1.4168$  &  $y(0.40) = 2.2012$  provided by the analytical solution  $y_{Exact} = 3(e^{2x} - e^x)$

III. Solve  $y' = \frac{dy}{dx} = x - y$  at  $x = 0.50, x = 0.70, x = 1.00, x = 1.30, x = 1.60$

With initial values  $y(0.00) = 0.000000, y(0.10) = 0.004837, y(0.20) = 0.018731, y(0.30) = 0.040818$  &  $y(0.40) = 0.070320$  provided by the analytical solution  $y_{Exact} = x - 1 + e^{-x}$

IV. Solve  $y' = \frac{dy}{dx} = \frac{1-y}{x}$  at  $x = 1.50, x = 1.70, x = 2.00, x = 2.30, x = 2.60$

With initial values  $y(1.00) = 2.00000000, y(1.10) = 1.90909100, y(1.20) = 1.83333000, y(1.30) = 1.76923077$  &  $y(1.40) = 1.71428570$  provided by the analytical solution  $y_{Exact} = 1 + \frac{1}{x}$

V. Solve  $y' = \frac{dy}{dx} = \frac{1}{2}(x + y)$  at  $x = 2.50, x = 3.50, x = 5.00, x = 6.50, x = 8.00$

With initial values  $y(0.00) = 2.0000, y(0.50) = 2.6361, y(1.00) = 3.5949, y(1.50) = 4.9680$  &  $y(2.00) = 6.8731$  provided by the analytical solution  $y_{Exact} = 4e^{x/2} - x - 2$

VI. Solve  $y' = \frac{dy}{dx} = x^2 + y$  at  $x = 1.25, x = 1.75, x = 2.50, x = 3.25, x = 4.00$

With initial values  $y(0.00) = 0.0000, y(0.25) = 0.0056, y(0.50) = 0.0474, y(0.75) = 0.1715$  &  $y(1.00) = 0.4366$  provided by the analytical solution  $y_{Exact} = 2(e^x - x - 1) - x^2$

### 5. Quantitative comparison of numerical results

Here we compare different estimated values obtained using the derived formulae with the exact values, and hence calculate the relative errors [1,5]. Here relative errors are given as percentages.

Ex-I. $n = 4, h = 0.10$					
$k$	Base	$x$	$y_{n+k}$	$y_{Exact}$	$ E_R $
1	$y_0$	0.50	2.255014	2.255252	0.01054%
3		0.70	2.510957	2.510338	0.02466%
6		1.00	3.093698	3.086161	0.24420%
9		1.30	3.978440	3.941829	0.92880%
12		1.60	5.272670	5.154929	2.28405%
1	$y_2$	0.50	2.255248	2.255252	0.00015%
3		0.70	2.510967	2.510338	0.02507%
6		1.00	3.093708	3.086161	0.24454%
9		1.30	3.978459	3.941829	0.92927%
12		1.60	5.272683	5.154929	2.28430%
1	$y_4$	0.50	2.255284	2.255252	0.00141%
3		0.70	2.511003	2.510338	0.02648%
6		1.00	3.093744	3.086161	0.24569%
9		1.30	3.978494	3.941829	0.93017%
12		1.60	5.272718	5.154929	2.28498%
Ex-II. $n = 4, h = 0.10$					
$k$	Base	$x$	$y_{n+k}$	$y_{Exact}$	$ E_R $
1	$y_0$	0.50	3.208665	3.208682	0.00052%
3		0.70	6.119890	6.124342	0.07269%
6		1.00	13.902437	14.012323	0.78421%
9		1.30	28.484602	29.383324	3.05861%
12		1.60	54.289384	58.738493	7.57443%
1	$y_2$	0.50	3.208668	3.208682	0.00061%
3		0.70	6.119887	6.124342	0.07274%
6		1.00	13.902434	14.012323	0.78423%
9		1.30	28.484599	29.383324	3.05862%
12		1.60	54.289380	58.738493	7.57444%
1	$y_4$	0.50	3.208628	3.208682	0.00168%
3		0.70	6.119853	6.124342	0.07330%
6		1.00	13.902400	14.012323	0.78447%
9		1.30	28.484565	29.383324	3.05874%
12		1.60	54.289346	58.738493	7.57450%
Ex-III. $n = 4, h = 0.10$					
$k$	Base	$x$	$y_{n+k}$	$y_{Exact}$	$ E_R $
1	$y_0$	0.50	0.106530	0.106531	0.00004%
3		0.70	0.196566	0.196585	0.00967%
6		1.00	0.367549	0.367879	0.08970%
9		1.30	0.570410	0.572532	0.37070%
12		1.60	0.793366	0.801897	1.06402%
1	$y_2$	0.50	0.106531	0.106531	0.00019%
3		0.70	0.196567	0.196585	0.00916%

6	$y_4$	1.00	0.367550	0.367879	0.08954%
9		1.30	0.570410	0.572532	0.37065%
12		1.60	0.793366	0.801897	1.06399%
1		0.50	0.106530	0.106531	0.00039%
3		0.70	0.196566	0.196585	0.00967%
6		1.00	0.367549	0.367879	0.08970%
9		1.30	0.570410	0.572532	0.37069%
12		1.60	0.793365	0.801897	1.06400%

Ex-IV. $n = 4, h = 0.10$					
$k$	Base	$x$	$y_{n+k}$	$y_{Exact}$	$ E_R $
1	$y_0$	1.50	1.666612	1.666667	0.00326%
3		1.70	1.585932	1.588235	0.14500%
6		2.00	1.459417	1.500000	2.70551%
9		2.30	1.181583	1.434783	17.64726%
12		2.60	0.408175	1.384615	70.52070%
1	$y_2$	1.50	1.666614	1.666667	0.00316%
3		1.70	1.585934	1.588235	0.14488%
6		2.00	1.459419	1.500000	2.70542%
9		2.30	1.181584	1.434783	17.64717%
12		2.60	0.408176	1.384615	70.25061%
1	$y_4$	1.50	1.666611	1.666667	0.00334%
3		1.70	1.585931	1.588235	0.14507%
6		2.00	1.459416	1.500000	2.70559%
9		2.30	1.181582	1.434783	17.64735%
12		2.60	0.408174	1.384615	70.52079%

Ex-V. $n = 4, h = 0.50$					
$k$	Base	$x$	$y_{n+k}$	$y_{Exact}$	$ E_R $
1	$y_0$	2.50	9.460738	9.461372	0.00670%
3		3.50	17.487774	17.518411	0.17489%
6		5.00	41.011533	41.729976	1.72165%
9		6.50	88.737632	94.661360	6.25781%
12		8.00	174.30669	208.392600	14.43713%
1	$y_2$	2.50	9.460799	9.461372	0.00606%
3		3.50	17.487835	17.518411	0.17454%
6		5.00	41.011593	41.729976	1.72150%
9		6.50	88.73769	94.661360	6.25775%
12		8.00	174.30675	208.392600	14.43710%
1	$y_4$	2.50	9.460768	9.461372	0.00638%
3		3.50	17.487804	17.518411	0.17471%
6		5.00	41.011563	41.729976	1.72158%
9		6.50	88.737662	94.661360	6.25778%
12		8.00	178.30672	208.392600	14.43712%

Ex-VI. $n = 4, h = 0.25$					
$k$	Base	$x$	$y_{n+k}$	$y_{Exact}$	$ E_R $
1	$y_0$	1.25	0.917885	0.918186	0.03278%
3		1.75	2.929791	2.946705	0.57400%
6		2.50	10.731895	11.114988	3.44663%
9		3.25	29.426511	32.518180	9.50751%
12		4.00	67.699540	83.196300	18.62674%
1	$y_2$	1.25	0.917816	0.918186	0.04030%
3		1.75	2.929723	2.946705	0.57630%
6		2.50	10.731884	11.114988	3.44673%
9		3.25	29.426443	32.518180	9.50772%
12		4.00	67.699472	83.196300	18.62682%
1	$y_4$	1.25	0.917865	0.918186	0.03496%

3		1.75	2.929771	2.946705	0.57468%
6		2.50	10.731875	11.114988	3.44681%
9		3.25	29.426491	32.518180	9.50757%
12		4.00	67.699520	83.196300	18.62677%

## 6. Conclusions

By observing above comparative tables, we have found that the derived extrapolation formulae provide very good accuracy up to 13<sup>th</sup> approximation of entry (value of dependent variable) for smaller values of  $h$  (spacing in independent variable). The fluctuation within exact and approximate values increased with the value of  $k$  (number of approximation), as shown by the relative errors. The fluctuation rose rapidly for the higher values of  $h$ . Again, the fluctuation occurred much for the differential equations having non-linear solutions; such as exponential, logarithmic and trigonometric functions. Moreover, we have seen that differential equations having solutions not passing through the origin differ in high rate within exact and approximate values.

Also, by comparing a previous work [8], it can be said that present work provided better accuracy and it can be improved more by further efficient choice of parameters.

Thus, we can conclude that, the derived extrapolation formulae will efficient for a long range of  $k$  if the solution curves of differential equations are linear and spacing in arguments (value of dependent variable) are small enough.

## 7. References

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